APPENDIX A: PSF Fit-Function Relations

A.1 MOFFIT FUNCTION FORMULAS

Basic Relations

The "Moffit" function is given as

$$M(r) = \frac{A}{[1 + (r/r_0)^2]^B} \tag{A1}$$

where

A = the central peak value (DN/pixel^2 or DN/arcsec^2)
r_0 = the radial scale factor of the core of the profile (pixel or arcsec)
B = the power law exponent regulating the scattering wings of the profile

The total DN encircled to radius r is

$$E_M(r) = 2\pi \int_0^r rM(r)dr = \frac{\pi A r_0^{(2B)}}{B-1}\left[\frac{1}{r_0^{2(B-1)}} - \frac{1}{\left(r_0^2 + r^2\right)^{(B-1)}}\right] \tag{A2}$$

The total DN to infinity is

$$E_\infty = \pi A r_0^2/(B-1) \tag{A3}$$

The fraction of energy encircled out to radius r is

$$E_{frac} = E_M(r)/E_\infty = 1 - \left[\frac{r_0^2}{\left(r_0^2 + r^2\right)}\right]^{(B-1)} = 1 - \left[\frac{1}{1 + \frac{r^2}{r_0^2}}\right]^{(B-1)} \tag{A4}$$

The radius at which the intensity falls to 1/2 the peak value is

$$r_{1/2} = r_0\sqrt{2^{1/B} - 1} \quad , \tag{A5}$$

and half the total energy is contained within a radius

$$r_{E_M/2} = r_0\sqrt{2^{1/(B-1)} - 1} \quad . \tag{A6}$$

Note that any given $r_{1/2}$ or $r_{E_M/2}$ can be obtained by suitable covariation of r_0, B, since the two are inversely related.

For comparison, a gaussian [$G = A\exp(-r^2/2\sigma^2)$] with width, σ, has half-maximum and half-power radii given by

$$r_{1/2} = r_{E_G/2} = 1.177\sigma \tag{A7}$$

Fit Parameter Dependencies

The derivatives of M with respect to the fit parameters $\{A, r_0, B\}$ play a role in the least-squares fitting routines and in the error analysis. For the Moffit fit, they are:

$$\begin{aligned}
\frac{\partial M}{\partial A} &= \frac{M}{A} \\
\frac{\partial M}{\partial r_0} &= \frac{2MBr^2}{r_0^3} \Big/ \left(1 + \frac{r^2}{r_0^2}\right) \qquad (A8) \\
\frac{\partial M}{\partial B} &= -M \log\left(1 + \frac{r^2}{r_0^2}\right)
\end{aligned}$$

A.2 POWER-LAW (HALO) FORMULAS

Basic Relations

The function for the extended power-law (halo) region is given as

$$P(r) = \frac{P_0}{(1+r)^D} \tag{A9}$$

where

P_0 = the central peak value
D = the power-law exponent

The distance r is given in arsec (pixel), and P_0 is stated in DN/arcsec^2 (DN/pixel^2).

The total DN encircled between radii R_{bp} and R_{P2} is

$$\begin{aligned}
E_P(r) &= 2\pi \int_{R_{bp}}^{R_{P2}} rP(r)dr \\
&= 2\pi P_0 \left[\frac{1}{(D-1)(1+r)^{D-1}} - \frac{1}{(D-2)(1+r)^{D-2}}\right]_{R_{bp}}^{R_{P2}}
\end{aligned} \tag{A10}$$

The total DN to infinity is unbounded for $D \le 2$.

Fit Parameter Dependencies

The derivatives of P with respect to the fit parameters $\{P_0, D\}$ are:

$$\frac{\partial P}{\partial P_0} = P/D$$
$$\frac{\partial P}{\partial D} = -P\log(1+r) \tag{A11}$$

A.3 UNCERTAINTY ANALYSIS

The uncertainties $\{\delta A, \delta r_0, \delta B, \delta P_0, \delta D\}$ in the fit parameters $\{A, r_0, B, P_0, D\}$ are determined by standard means in the IDL least-squares package `CURVEFIT`.

The uncertainty in compound parameters, like $r_{1/2}$, is computed as follows. First, take partial derivatives with respect to the fit parameters:

$$\frac{\partial r_{1/2}}{\partial r_0} = \sqrt{2^{1/B}-1} = \frac{r_{1/2}}{r_0}$$

$$\frac{\partial r_{1/2}}{\partial B} = -\frac{1}{2}\frac{r_0^2}{r_{1/2}}\frac{\ln 2}{B^2}$$

Then the uncertainty $\delta r_{1/2}$ in $r_{1/2}$ is given by

$$\delta r_{1/2} = \sqrt{\left(\frac{\partial r_{1/2}}{\partial r_0}\delta r_0\right)^2 + \left(\frac{\partial r_{1/2}}{\partial B}\delta B\right)^2} \tag{A12}$$

Similarly, the error in the total DN in an image can be assessed from the integral relations (A2) and (A10). The DN in the Moffit domain out to radius r_m arcsec from point-image center is given by:

$$T_m(r) = 2\pi\int_0^{r_m} \frac{A}{[1+(r/r_0)^2]^B}\; rdr \tag{A13a}$$

where the central peak value, A, is given in DN/arcsec^2. Likewise, the total DN between r_1 and r_2 for the power law part of the fit is:

$$T_p(r) = 2\pi\int_{r_1}^{r_2} \frac{P_0}{(1+r)^D}\; rdr \tag{A13b}$$

where P_0 is again in DN/arcsec^2. For this analysis, where the two functions are spliced together piecewise, the parameters $r_m = r_1 = R_{bp}$ and $r_2 = R_{P2} = 630$ arcsec.

With the fit parameter uncertainties given by `CURVEFIT` as $\{\delta A, \delta r_0, \delta B, \delta P_0, \delta D\}$, the formal error in the total DN is:

$$\Delta_{\mathrm{DN}} = \sqrt{\left(\frac{\partial T_m}{\partial A}\delta A\right)^2 + \left(\frac{\partial T_m}{\partial r_0}\delta r_0\right)^2 + \left(\frac{\partial T_m}{\partial B}\delta B\right)^2 + \left(\frac{\partial T_p}{\partial P_0}\delta P_0\right)^2 + \left(\frac{\partial T_p}{\partial D}\delta D\right)^2} \ . \tag{A14}$$

The component error dependencies for (A14) are:

$$\frac{\partial T_m}{\partial A} = \frac{r_0^{2B}}{2(B-1)}\ \left[r_0^{2(1-B)} - x\right]$$

$$\frac{\partial T_m}{\partial r_0} = \frac{A}{(B-1)}\ \left[(B-1)r_0^{(1+2B)}y^{-B} - Bxr_0^{(2B-1)} + r_0\right]$$

$$\frac{\partial T_m}{\partial B} = \frac{A}{2(B-1)^2}\left[xr_0^{2B}\left((B-1)\log\left(y/r_0^2\right)+1\right) - r_0^2\right]$$

where

$$y = (r_0^2 + r_m^2) \quad , \quad x = y^{(1-B)} \quad ,$$

and

$$\frac{\partial T_p}{\partial P_0} = \frac{1}{(D-1)(D-2)} * \\ \Big[\ (1+r_1)^{-D}\Big(r_1^2(D-1) + r_1 D + 1\Big) \\ -\ (1+r_2)^{-D}\Big(r_2^2(D-1) + r_2 D + 1\Big)\ \Big]$$

$$\frac{\partial T_p}{\partial D} = \frac{P_0}{(D-1)(D-2)} * \\ \Big[\ (1+r_2)^{-D}\Big(q_2(\log(1+r_2) + \frac{1}{(D-2)} + \frac{1}{(D-1)}) - r_2(r_2+1)\Big) \\ -\ (1+r_1)^{-D}\Big(q_1(\log(1+r_1) + \frac{1}{(D-2)} + \frac{1}{(D-1)}) - r_1(r_1+1)\Big)\ \Big]$$

where

$$q_1 = r_1^2(D-1) + r_1 D + 1 \quad , \quad q_2 = r_2^2(D-1) + r_2 D + 1 \quad .$$